Approximation Algorithms for Cutting a Convex Polyhedron Out of a Sphere

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Abstract. This paper presents the following approximation algorithms for computing a minimum cost sequence of planes to cut a convex polyhedron P of n vertices out of a sphere Q: an $O(n \log n)$ time $O(\log^2 n)$-factor approximation, an $O(n^{1.5} \log n)$ time $O(\log n)$-factor approximation, and an $O(1)$-factor approximation with exponential running time. Our results significantly improve upon the previous $O(n^3)$ time $O(\log^2 n)$-factor approximation solution.

1 Introduction

About two and a half decades ago, Overmars and Welzl considered the following problem: Given a polygonal piece of paper Q with a polygon P of n vertices drawn on it, cut P out of Q by a sequence of "guillotine cuts" in the cheapest possible way [9]. After the hardness of computing an optimal cutting sequence was shown by Bhadury and Chandrasekaran [3], the research has recently been concentrated on finding approximation solutions. Particularly, when both P and Q are convex polygons in the plane, several $O(\log n)$ and constant factor approximation algorithms and a PTAS have been proposed [2,4,5,10]. The study of this type of problems is mainly motivated by the application where a given shape needs to be cut out from a parent of material.

In three dimensions, Jaromczyk and Kowaluk have studied the problem of cutting polyhedral shapes with a hot wire cut and give an $O(n^5)$ time algorithm that constructs a cutting path, if it exists [8]. Very recently, S. I. Ahmed et al. considered the following problem in three dimensions: Given a convex polyhedron P of n vertices inside a sphere Q, the objective is to compute a minimum cost sequence of planes to cut Q such that after the last cut of the sequence we have $Q = P$ [1]. Here, the cost of a plane cut is the area of the intersection of the plane with the current polyhedron Q. Their proposed algorithm runs in $O(n^3)$ time and has the cutting cost $O(\log^2 n)$ times the optimal. Whether the approximation factor or the time complexity can be improved is left as an open problem.

In this paper, we present three approximation algorithms for finding a minimum cost sequence of planes to cut P out of Q: an $O(n \log n)$ time $O(\log^2 n)$-factor approximation, an $O(n^{1.5} \log n)$ time $O(\log n)$-factor approximation, and

M. Atallah, X.-Y. Li, and B. Zhu (Eds.): FAW-AAIM 2011, LNCS 6681, pp. 125–135, 2011.

an $O(1)$-factor approximation with exponential running time. This gives a significant improvement upon the previous result [1]. Our algorithms make use of graph-decompositions based on the planar separator theorem [7], and extend some known planar frameworks to three dimensions.

2 Preliminaries

Assume that a convex polyhedron P of n vertices is completely contained in a sphere Q. A *guillotine cut*, or simply a *plane cut* is a cut that does not intersect the interior of P and divides Q into two convex pieces, lying on both sides of the cut. Particularly, a plane cut is a *face/edge/vertex cut* if it cuts along a face/edge/vertex of P. After a cut is made, Q is updated to the piece containing P. A *cutting sequence* is a sequence of plane cuts such that after the last cut in the sequence we have $P = Q$.

The cost of a cut is the area of the intersection of the cut with Q. Our objective is to find a cutting sequence whose total cost is minimum. Let us denote by $|S|$ the cost of a cutting sequence S. An *optimal* cutting sequence S^* is a cutting sequence whose cost $|S^*|$ is minimum. A *face* cutting sequence, denoted by S_f, is a sequence of plane cuts that are all made along the faces of P. An optimal face cutting sequence S_f^* is a face cutting sequence whose cost $|S_f^*|$ is minimum among all face cutting sequences.

Let f denote a face of the polyhedron P. We will denote by $|f|$ the area of f. Also, we let $|P| = \sum |f|$, for all $f \in P$, i.e., $|P|$ denotes the surface area of P. For two points x, y, we denote by xy the line segment connecting x and y, and denote by $|xy|$ the length of the segment xy.

2.1 Lower Bounds

In order to estimate the cost performance of our approximation algorithms given in the next two sections, we now establish some lower bounds on $|S^*|$. A trivial lower bound on $|S^*|$ is clearly $|P|$. The following two lower bounds are similar to, but slightly different from, those of [1].

Lemma 1. *Suppose that the center o of the sphere Q is contained in the polyhedron P. Let f denote the face of P such that the distance between the supporting plane of f and the center o is minimum, and let R_1 denote the radius of the intersection (circle) of Q with the supporting plane of f. Then, $|S^*| \geq \frac{\pi}{2} R_1^2$.*

Proof. First, an optimal cutting sequence contains a face cut along f; otherwise, P cannot be cut out, a contradiction. Assume below that the kth cut in $S^*(= C_1, C_2, \ldots, C_k, \ldots)$ is made along the face f.

We claim that the total cost of $C_1, C_2, \ldots, C_k$ is at least $\frac{\pi}{2} R_1^2$. It is trivially true when $k = 1$. Assume below that $k > 1$. Denote by H_1 the intersection (circle) of the sphere Q with the supporting plane of f. Let A be the portion of H_1, which is cut out by C_k. See Fig. 1. Clearly, $|A| = |C_k|$. Denote by D and E the portions of H_1 which are to the left and the right of A, respectively. Without

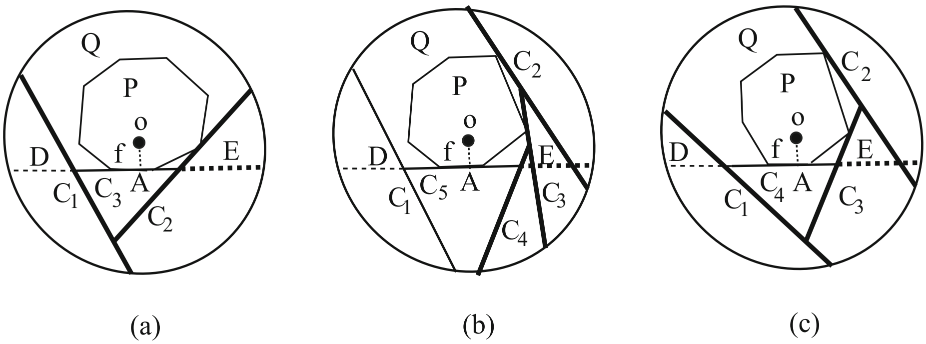

Fig. 1. A planar view of several cuts, with the last one made along the face f

loss of generality, assume that neither D nor E is empty. Assume that C_i and C_j $(1 \leq i, j \leq k-1)$ are the last cuts that cut off D and E, respectively. Also, assume $i < j$.

Consider first the simplest situation in which $i = 1$ and $j = 2$. See Fig. 1(a). Since D is contained in the portion of the given sphere Q, which is cut off by C_1, we have $|D| \leq |C_1|$. If C_2 does not intersect C_1, we also have $|E| \leq |C_2|$, and thus $2|S^*| > |C_1| + |C_2| + |C_3| \geq |D| + |E| + |A| = \pi R_1^2$. The claim is true. In the case that C_2 intersects with C_1, E is contained in the portion of the given or original sphere, which is cut off by C_1 and C_2. See Fig. 1(a) for an example, where C_1, C_2 and E are drawn in bold line. We then have $|E| \leq |C_1| + |C_2|$. Hence, $2|S^*| > |C_1| + (|C_1| + |C_2|) + |C_3| \geq |D| + |E| + |A| = \pi R_1^2$, and the claim also holds.

Consider next the situation in which $i = 1$ and $j > 2$. Note that $C_2, \ldots, C_j$ cut out an outward-convex surface on the resulting polygon Q. If none of $C_2, \ldots, C_j$ intersects with C_1, then E is contained in the portion of the given sphere, which is cut off by $C_2, \ldots, C_j$ (Fig. 1(b)). Otherwise, E is contained in the portion cut off by $C_1, C_2, \ldots, C_j$ (Fig. 1(c)). In either case, we have $|E| \leq |C_1| + |C_2| + \ldots + |C_j|$, and thus $2|S^*| \geq \pi R_1^2$. Our claim is true, again.

For the last situation in which $i > 1$ and $i < j$, we can also see that D (E) is contained the portion of the given or original sphere, which is cut off by the cuts $C_1, \ldots, C_i$ (the cuts $C_1, \ldots, C_j$). So we have $|C_1| + \ldots |C_i| \geq |D|$, $|C_1| + \ldots |C_j| \geq |E|$, and $2|S^*| \geq \pi R_1^2$. Our claim is thus proved, and the lemma follows. □

Remark 1. In Lemma 3 of [1], it was claimed that $|S^*| \geq \pi R^2$, where R denotes the radius of the given sphere Q. Only a short sketch was described, in which two special cases (i.e., P denegerates to a point or the sphere Q itself) are considered [1]. A more strict proof should further be given.

Lemma 2. *Suppose that the center o of Q is not contained in P. Let p denote the point of P which is closest to o, and let R_2 denote the radius of the intersection (circle) of Q with the plane, which is perpendicular to the segment op at the point p. Then, $|S^*| \geq \frac{\pi}{2} R_2^2$.*

Proof. Denote by H_2 the intersection (circle) of the sphere Q with the plane, which is perpendicular to the segment *op* at the point p. Clearly, H_2 touches a face/edge/vertex of P, and does not intersect P. To cut out P from Q, the circle H_2 has to be cut out, too. By an argument similar to the proof of Lemma 1, we can then obtain $|S^*| \geq \frac{\pi}{2} R_2^2$. □

3 An Efficient $O(n \log n)$ Time Approximation Algorithm

As in the previous work [1,5], our algorithm consists of two phases: *box cutting phase* and *carving phase*. In the box cutting phase, we cut a bounding box B out of Q such that P is contained in B. (Note that B is used for the worst case analysis, and only part of the box B may actually result.) Then in the carving phase, we further cut P out of B.

The following new ideas allow us to develop an efficient $O(n \log n)$ time approximation solution. Instead of finding a minimum box bounding P used in [1], we present a simple linear-time algorithm to compute a bounding box B, with $|B| \leq 6|P|$. In the carving phase. we employ the planar separation theorem, so as to accelerate the process of cutting P out of B.

3.1 Box Cutting Phase

In this section, we present a linear-time algorithm for finding a bounding box B of P with $|B| \leq 6|P|$, and then, we show how to cut B out of Q at cost $O(|S^*|)$.

Lemma 3. *For a convex polyhedron P of n vertices, one can compute in $O(n)$ time a bounding box B such that P is contained in B, with $|B| \leq 6|P|$.*

Proof. First, we find two vertices s, t of P such that the z-coordinates of s and t are minimum and maximum, respectively. (Note that two points s, t do not give the diameter of the set of all vertices of P.) Without loss of generality, assume that st is parallel to the z-axis; otherwise, we can simply rotate the coordinate axes such that the z-axis is parallel to st. Clearly, it takes $O(n)$ time to compute s and t.

Now, we project all vertices of P into the (x, y) plane vertically. Denote by P_2 the set of the projected vertices of P in the (x, y) plane. Next, we compute two points u, v of P_2 such that the x-coordinates of u and v are minimum and maximum, respectively. Again, we can assume that uv is parallel to the x-axis. Finally, compute two vertices u', v' of P_2 such that their y-coordinates, denoted by $y(u')$ and $y(v')$, are minimum and maximum, respectively. See Fig. 2. (We cannot assume that $u'v'$ is parallel to the y-axis. However, the value $y(v') - y(u')$ is sufficient for the performance analysis of our algorithm.)

Let B be the minimum axis-aligned box that encloses P. As discussed above, the lengths of B in three different directions are $|st|$, $|uv|$ and $y(v') - y(u')$. (Remember that s and t (u and v) are two vertices of P (P_2).) We show below that $|B| \leq 6|P|$. Denote by $CH(P_2)$ the convex hull of the point set P_2 in the (x, y) plane, and denote by $|CH(P_2)|$ the area of $CH(P_2)$. Clearly,

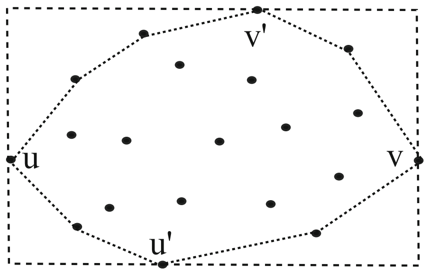

Fig. 2. The orthogonal projection of the vertices of P in the (x, y) plane

$|P| \geq 2|CH(P_2)|$. (Note that we needn't compute $CH(P_2)$ at all.) Since the segment uv is parallel to the x-axis, we have $|CH(P_2)| \geq (y(v') - y(u'))|uv|/2$. Thus, $|P| \geq (y(v') - y(u'))|uv|$. Since st is parallel to the z-axis, we can similarly obtain $|P| \geq (y(v') - y(u'))|st|$ and $|P| \geq |uv| \cdot |st|$. In summary, we have

$$|B| = 2((y(v') - y(u'))|uv| + (y(v') - y(u'))|st| + |uv| \cdot |st|) \leq 6|P|,$$

as required. □

Lemma 4. *For a convex polyhedron P of n vertices inside the sphere Q, one can compute in $O(n)$ time a cutting sequence of cost $O(|S^*|)$, which cuts the box B (bounding P) out of Q.*

Proof. We mainly distinguish two different situations. If the center o of Q is contained in the polyhedron P, then we simply make six cuts along all faces of the box B. From the definition of the radius R_1 (see Lemma 1), each of these cuts is of cost no more than πR_1^2. Thus, the cost of this cutting sequence is at most $12|S^*|$. In the case that the center o of Q is not contained in P, we first make a cut along the plane, which is perpendicular to the segment op at p, where p is the point of P that is closest to the center o of Q. Following from Lemma 2, its cutting cost πR_2^2 is no more than $2|S^*|$. For the remaining part of the sphere (that contains P), we further make six cuts along all faces of the box B. Since any of these six cuts is of cost at most πR_2^2, the cost of this cutting sequence is at most $14|S^*|$.

Consider now the time required to compute the cutting sequence described above. Since P is convex, whether the center o of Q is contained in P can simply be determined in $O(n)$ time. If o is not contained in P, we further compute in $O(n)$ time the point p of P that is closest to o. It gives the first cutting plane that is perpendicular to op in this case. Since finding the bounding box B of P also takes $O(n)$ time, the proof is thus complete. □

3.2 Carving Phase

Denote by P^+ (P^-) the set of the faces of P, whose outward normal has the positive (negative) z value. We describe below how to cut all faces of P^+ out of the box B. (Cutting P^- out of B can be done analogously.) Since P is convex, P^+ is clearly a planar graph.

An important observation made here is that P^+ can be cut out from the box B using the *planar separator theorem* [7]. Suppose first that each face of P^+ has been triangulated. Consider the dual graph of the triangulation of P^+, which has $O(n)$ edges and nodes. We can select in linear time $O(\sqrt{n})$ edges to form a separator $\mathcal{T}$, which partitions the (dual) graph into two portions with at most two third of the edges on each side of $\mathcal{T}$ [7]. Our idea is then to perform a sequence of plane cuts along all faces of $\mathcal{T}$. (Note that each node of the separator $\mathcal{T}$ corresponds to a triangle or face of P^+.) So $P+$ can be cut out from B using a divide-and-conquer procedure. For each separator $\mathcal{T}$, the other divide-and-conquer procedure will also be used to give the face cutting sequence along $\mathcal{T}$, which total cost is $O(|S^*| \cdot \log n)$. For this purpose, we order all nodes of $\mathcal{T}$ from one of its ends to the other, and define the *median* node of $\mathcal{T}$ to be the node of $\mathcal{T}$ with the middle index.

Lemma 5. *The convex polyhedron P can be cut out from the box B in $O(n \log n)$ time by a cutting sequence of cost $O(|S^*| \cdot \log^2 n)$.*

Proof. First, we triangulate each face of P^+. Consider the dual graph of the triangulation of P^+, which has $O(n)$ edges and nodes. We can then select in linear time $O(\sqrt{n})$ edges to form a separator $\mathcal{T}$, which partitions the (dual) graph into two portions with at most two third of the edges on each side of $\mathcal{T}$.

Let us now consider how to perform a sequence of plane cuts along the found separator $\mathcal{T}$. We use $O(\log n)$ recursive steps. In the first step, we find the median node of $\mathcal{T}$, and make a plane cut C_1 along its corresponding triangle or face of P^+. Clearly, $|C_1| \leq |B|$, and the cut C_1 divides the separator $\mathcal{T}$ into two sub-separators. (Note that C_1 actually contains all triangles that are on the same face as the chosen triangle.) In the next step, we find the median node in each of the sub-separators, and make two cuts C_2, C_3. Note that $|C_2| + |C_3| \leq |B|$. This operation is performed until all plance cuts along the nodes of $\mathcal{T}$ are made. In each recursive step, the cutting cost is no more than $|B|$. Therefore, the total cost taken for the face cutting sequence along $\mathcal{T}$ is $O(|B| \cdot \log n) = O(|S^*| \cdot \log n)$.

After the cutting sequence along the separator $\mathcal{T}$ is made, the problem of cutting out the faces of P^+ is partitioned into two subproblems; one on each side of $\mathcal{T}$. Denote by B_1, B_2 the two portions (of the original box B), which is obtained after the cutting sequence along $\mathcal{T}$ is made, on different sides of $\mathcal{T}$. Thus, $|B_1| + |B_2| \leq |B|$. Next, we further apply the planar separator theorem to B_1, B_2, and denote by $\mathcal{T}_1$, $\mathcal{T}_2$ the found planar separators, respectively. Then, perform the cutting sequences along $\mathcal{T}_1$ and $\mathcal{T}_2$, separately. Again, the cutting costs are $O(|B_1| \cdot \log n)$ and $O(|B_2| \cdot \log n)$, respectively. So the cutting cost taken in the second step of our algorithm is also $O(|S^*| \cdot \log n)$. In this way, all faces of P^+ can be cut out in at most $O(\log n)$ recursive steps. Since the cutting cost of each recursive step is $O(|S^*| \cdot \log n)$, the total cost taken by our algorithm is $O(|S^*| \cdot \log^2 n)$.

Finally, since the separator of a planar graph can be found in linear time, the total time required to cut P out of B is $O(n \log n)$. □

The first result of this paper immediately follows from Lemmas 3 to 5.

Theorem 1. *For a given convex polyhedron P of n vertices inside a sphere Q, an $O(\log^2 n)$-factor approximation of an optimal cutting sequence for cutting P out of Q can be computed in $O(n \log n)$ time.*

4 Constant and $O(\log n)$ Factor Approximation Algorithms

To obtain a good approximation of an optimal cutting sequence, we will extend some known planar frameworks to three dimensions. First, we show a general property of an optimal cutting sequence S^*, i.e., any cut of S^* has to touch P. Next, we present a constant factor and an $O(\log n)$ factor approximation algorithms for cutting P out of Q.

4.1 A General Property

Assume that both Q and P are the convex polyhedra. Then, we have the following result.

Lemma 6. *Any cut of an optimal cutting sequence S^* for cutting P out of Q has to touch a vertex, an edge or a face of P.*

Proof. The proof is by contradiction. Suppose that C^* is the first cut of S^*, which does not touch P. Clearly, moving or deleting C^* does not change the cost of the cuts before C^*. If no cuts after C^* end on the cut C^*, then C^* can be deleted from S^* without increasing the cost of any other cuts; it contradicts the optimality of S^*. Denote by X the set of the cuts after C^*, which end on the cut C^*. Clearly, moving C^* will change the cost of the cuts of X.

Assume first that C^* does not contain any vertex of the current polyhedron Q. Denote by C_1, C_2 the two cuts which are obtained by moving C^* parallel to itself, towards and away from P, by a very small distance ϵ such that the shape of C_1 (C_2) on the surface of Q is similar to that of C^*, if the cut C^* of S^* is replaced by C_1 (C_2). Since ϵ is arbitrarily small, neither C_1 nor C_2 touches P. Then, $|S^*|$ cannot be strictly smaller than the cost of either *new* sequence (i.e., $|S^*|$ is equal to the cost of both new sequences), in which the cut C^* of S^* is replaced by C_1 or C_2; otherwise, due to the similarity, the cost of one new cutting sequence is strictly smaller than $|S^*|$, a contradiction. Assume now that $|S^*|$ is equal to the cost of both new sequences. Remember that the current polyhedron Q containing P is always convex. So if we keep to move C^* *away from* P, by a distance ϵ every time, the change in the total area of the current cut C^* and all the cuts of X, which still end on the current cut C^*, is a monotone decreasing function. This implies that either a new position of C^* yields an empty set X, or C^* is eventually moved outside the polyhedron Q; a contradiction occurs in either case.

It requires a little more care of the situation in which C^* contains at least one vertex of the current polyhedron Q. Also, denote by C_1 (C_2) the cut, which is obtained by moving C^* parallel to itself, towards (away from) P, by a very small distance ϵ. Since C_1 is not similar to C_2 in this case, it needs a slightly different argument. Suppose now that we move the plane containing C^* *away from* P by the same distance ϵ, and consider the (convex) region C_2' in that plane, which is bounded by its intersection with the planes containing all faces of X. Note that C_2' is similar to both C^* and C_1. We call C_2' a *pseudo-cut*, and the cutting sequence in which C^* of S^* is replaced by C_2' and the cuts of X are extended to end on C_2' (if needed), a *pseudo-cutting sequence*. As discussed above, $|S^*|$ is then equal to the cost of of this pseudo-cutting sequence. It also follows from the convexity of Q and the above construction of the pseudo-cut C_2' that $C_2' \supseteq C_2$. Thus, the cost of the cutting sequence in which C^* of S^* is replaced by C_2, is no more than the cost of the above pseudo-cutting sequence. This implies that the operation of keeping to move C^* away from P also works, which eventually leads to a contradiction. The proof is thus complete. □

4.2 Algorithms in the Carving Phase

Suppose that the box B has been cut out from the sphere Q, as described in Section 3.1. We will focus our attention on the problem of cutting P out of B. For ease of presentation, we still use S^* to represent an optimal cutting sequence for cutting P out of B, and S_f^* an optimal face cutting sequence for cutting P out of B.

In the following, we first show that an optimal face cutting sequence for cutting P out of B is a constant factor approximation of an optimal cutting sequence. To obtain an $O(\log n)$-factor approximation, we further employ a dynamic programming technique, which was originally given by Overmars and Welzl [9].

Lemma 7. *In the carving phase of cutting P out of B, an optimal face cutting seqeunce S_f^* is an $O(1)$-factor approximation of S^*.*

Proof. Our proof is similar to that of its planar counterpart given by Daescu and Luo [4]. Let S^* be an optimal cutting seuqence for cutting P out of B. We will construct a face cutting sequence S_f, whose cost is at most $10|S^*|$. Since $|S_f^*| \leq |S_f|$ holds, the lemma then follows.

For every optimal cut $C^* \in S^*$, in order, if C^* is a face cut, we simply add it to S_f. Otherwise, C^* is tangent to a vertex or an edge of P; in this case, we add several face cuts to S_f as follows. If C^* is tangent to an edge e of P, then we add to S_f two cuts C_1, C_2 (in this order), which are made along the two faces of P having the common edge e. A portion of C_1 lies outside of C^* as viewed from a point inside P, but the whole cut C_2 lies inside of C^*. Since the original polyhedron containing P is the box B, the portion of C_1 lying outside C^* is of area at most $|C^*|$. If C^* touches a vertex v of P, we first project all the edges having the common vertex v into the plane C^* vertically, and find the two edges e_1, e_2 such that the smaller angle (less than π) between their projections in the plane C^* is maximum among all of these angles. Next, we add to S_f the cuts C_{11} and C_{12} (C_{21} and C_{22}), in this order, which are made along the two faces of P having the common edge e_1 (e_2). (Two of these faces may be identical.) Again, the portion of C_{11} (C_{21}) lying outside C^* is of area at most $|C^*|$.

Since some cuts of S^* may not be the face cuts, we give below a method to bound the extra cost between $|S_f|$ and $|S^*|$. Denote by B^* the portion of the original box B, which is obtained after C^* in the cutting sequence S^* is made, and B_f the portion of B obtained after the cuts corresponding to C^* in S_f are made. It follows from the above construction of S_f that $B_f \subseteq B^*$.

For a cut $C^* \in S^*$, at most four face cuts may have been added to S_f. As discussed above, the portions of these faces lying outside B^* are of area at most $2|C^*|$. So the extra cost taken for all such portions (lying outside B^*) is at most $2|S^*|$. Let us denote by C_1' and C_2' (C_{11}', C_{12}', C_{21}' and C_{22}') the portions of the cuts C_1 and C_2 (C_{11}, C_{12}, C_{21} and C_{22}), which are contained in B^*. Let Δ denote the part of B^*, which is exactly cut off by the cuts C_1 and C_2 (C_{11}, C_{12}, C_{21} and C_{22}). Since both B^* and P are convex, the *inner* surface of Δ, which consists of C_1' and C_2' (C_{11}', C_{12}', C_{21}' and C_{22}'), is inward-convex, and the *outer* surface of Δ, which consists of all other faces (including C^*) of Δ, is outward-convex. Therefore, the area of the inner surface of Δ is no more than that of the outer surface. (It can simply be proved by an argument similar to that given for its planar counterpart [6].) The extra cost between $|C^*|$ and $|C_1'|+|C_2'|$ ($|C_{11}'|+|C_{12}'|+|C_{21}'|+|C_{22}'|$) is thus bounded by the total area of the faces f of Δ, excluding C^*, C_1' and C_2' (C_{11}', C_{12}', C_{21}' and C_{22}'). Since these faces f belong to the surface of the original box B or some cuts of S^*, which are made before C^*, the extra cost taken for all the portions of C_1' and C_2' (C_{11}', C_{12}', C_{21}' and C_{22}') lying inside B^* is at most $|B| + |S^*|$. Putting together all results, we have $|S_f| \leq 2|S^*| + |S^*| + (|S^*| + |B|) \leq 4|S^*| + 6|P| \leq 10|S^*|$. This completes the proof. □

An optimal face cutting sequence S_f^* can be computed in exponential time, because the number of all face cutting sequences is trivially bounded by $n!$. So we have the following result.

Theorem 2. *For a given convex polyhedron P of n vertices inside a sphere Q, an $O(1)$-factor approximation of an optimal cutting sequence for cutting P out of Q can be computed in exponential time.*

It is not known whether there exists a polynomial time algorithm for computing S_f^*. However, optimal face cutting sequences for planar separators employed in the previous section can be computed using dynamic programming. This yields an $O(\log n)$-factor approximation algorithm for computing S_f^*.

Lemma 8. *The convex polyhedron P can be cut out from the box B in $O(n^{1.5} \log n)$ time by a face cutting sequence of cost $O(|S_f^*| \cdot \log n)$.*

Proof. As in the proof of Lemma 5, we employ a divide-and-conquer algorithm to cut P out of B. In every step of cutting along a separator, instead of a simple divide-and-conquer method, we use a dynamic programming algorithm to compute an optimal face cutting sequence for the separator. This allows us to cut P in $O(n^{1.5} \log n)$ time out of B at cost $O(|S_f^*| \cdot \log n)$.

The dynamic programming algorithm for plane cuts along a planar separator is essentially the same as that for line cuts along a convex chain in the plane [9].

Let $\mathcal{T}^1$ be the separator of P^+, which is found in the first step of our algorithm. Let us number the faces of $\mathcal{T}^1$ from 1 to m along the surface of P^+. (All triangles on the same plane are considered as a single face.) Then, we compute optimal face cutting sequences of all arc of faces of $\mathcal{T}^1$, in order of length of the arc, i.e., start with the arc of one face, next arcs of two arcs. Note first that the optimal face cutting sequence for the arcs having only one face f_i is the face f_i itself, as the cuts along f_{i-1} and f_{i+1} are assumed to have been made. (No face cut is made along f_{i-1} or f_{i+1} if $i = 1$ or $i = m$.) Assume now that all optimal face cutting sequences for the arcs of length no more than $j - i - 1$, $j > i$, have been computed. Consider how to cut out the faces between f_i and f_j. An optimal face cutting sequence consists of a cut along some face f_k $(i \le k \le j)$, followed by at most two optimal cutting sequences; one for all faces between f_i and f_{k-1} and the other for all faces between f_{k+1} and f_j. Trying all possible choices of k clearly gives an optimal face cutting sequence for all faces between f_i and f_j. Note that a face cut along f_k may intersect with the previously made cuts f_{i-1} and f_{j+1}, or the surface of the box B. After all the first two cuts along f_i and f_j $(1 \le i, j \le m)$ inside B have been precomputed in $O(m^2)$ time, such a try can be done in constant time. So after an $O(m^2)$-time preprocessing step, an optimal face cutting sequence for the arc between (any) two faces f_i, f_j can be computed in $O(j - i)$ time. Since the total number of these arcs is bounded by $O(m^2)$, and since m is of size $O(\sqrt{n})$, the time required to find an optimal cutting sequence along $\mathcal{T}^1$ is $O(m^3)$ or $O(n^{1.5})$.

An optimal face cutting sequence along the separator $\mathcal{T}^1$ divides the problem of cutting out the faces of P^+ into two independent subproblems; either subproblem is of size at most $2n/3$. Solving the recurrence $T(n) = 2T(2n/3) + O(n^{1.5})$ gives us the time bound $O(n^{1.5} \log n)$.

Next, we show that the cutting cost taken in each recursive step of our algorithm is $O(|S_f^*|)$. Let S_f^1 denote the optimal face cutting sequence along $\mathcal{T}^1$. Clearly, we have $|S_f^1| \le |S_f^*|$. Denote by B^1 the polyhedron obtained after S_f^1 is made. Denote by $\mathcal{T}^2$, $\mathcal{T}^3$ the two planar separators found in the second step of our divide-and-conquer algorithm, and S_f^2, S_f^3 the optimal face cutting sequences along $\mathcal{T}^2$, $\mathcal{T}^3$. The problem of cutting P^+ out of B^1 can then be considered as two independent subproblems, which are separated by $\mathcal{T}^1$. Since the face cutting sequence S_f^2 or S_f^3 is optimal only for the separator found in either side of $\mathcal{T}^1$, the cost $(|S_f^2| + |S_f^3|)$ is no more than the cost of an optimal face cutting sequence for cutting P^+ out of B^1. Moreover, since $B^1 \subset B$, the cost of any optimal face cutting sequence for cutting P^+ out of B^1 is no more than $|S_f^*|$. Thus, we have $|S_f^2| + |S_f^3| \le |S_f^*|$. Analogously, the cutting cost at each recursive step of our algorithm is no more than $|S_f^*|$. Since it needs at most $O(\log n)$ recursive steps, the total cost of our cutting sequence is $O(|S_f^*| \cdot \log n)$. □

The main result of this paper follows from Lemmas 4, 7 and 8.

Theorem 3. *For a given convex polyhedron P of n vertices inside a sphere Q, an $O(\log n)$-factor approximation of an optimal cutting sequence for cutting P out of Q can be computed in $O(n^{1.5} \log n)$ time.*

5 Concluding Remarks

We have presented three approximation algorithms for computing a minimum cost sequence of planes to cut a convex polyhedron P of n vertices out of a sphere Q. Our algorithms with $O(n^{1.5} \log n)$ running time $O(\log n)$-factor approximation and $O(n \log n)$ running time $O(\log^2 n)$-factor approximation greatly improve upon the previously known $O(n^3)$-time $O(\log^2 n)$-factor approximation solution.

Finally, we pose two open questions for further research. First, it is open to find a polynomial-time algorithm for computing an optimal face cutting sequence for cutting the convex polyhedron P out of the box B. Although its planar counterpart is true [9], we find it difficult to obtain the same result in 3D. Also, it is an interesting work to develop an approximation algorithm for cutting P out of another convex polyhedron Q. Again, whether the method used for its planar counterpart [5] can be generalized to 3D remains open.

Acknowledgements

We thank the anonymous referees for valuable comments. This work was supported by an oversea project of State Key Lab. for Novel Software Technology, Nanjing University, China.

References

1. Ahmed, S.I., Hasan, M., Islam, M.A.: Cutting a convex polyhedron out of a sphere. In: Rahman, M. S., Fujita, S. (eds.) WALCOM 2010. LNCS, vol. 5942, pp. 94–101. Springer, Heidelberg (2010)
2. Bereg, S., Daescu, O., Jiang, M.: A PTAS for cutting out polygons with lines. Algorithmica 53, 157–171 (2009)
3. Bhadury, J., Chandrasekaran, R.: Stock cutting to minimize cutting sequence. European Journal of Operational Research 88(1), 69–87 (1996)
4. Daescu, O., Luo, J.: Cutting out polygons with lines and rays. International Journal of Computational Geometry & Applications 16(2-3), 227–248 (2006)
5. Dumitrescu, A.: An approximation algorithm for cutting out convex polygons. Computational Geometry: Theory and Applications 29(3), 223–231 (2004)
6. Hershberger, J., Suri, S.: Practical methods for approximating shortest paths on a convex polytope in R^3. Computational Geometry: Theory and Applications 10, 31–46 (1998)
7. Lipton, R.J., Tarjan, R.E.: A separator theorem for planar graphs. SIAM J. Appl. Math. 36, 177–189 (1979)
8. Jaromczyk, J.W., Kowaluk, M.: Sets of lines and cutting out polyhedral objects. Computational Geometry: Theory and Applications 25, 67–95 (2003)
9. Overmars, M.H., Welzl, E.: The complexity of cutting paper. In: Proc. of the 1st Annual ACM Symposium on Computational Geometry, Maryland, USA, pp. 316–321 (1985)
10. Tan, X.: Approximation algorithms for cutting out polygons with lines and rays. In: Wang, L. (ed.) COCOON 2005. LNCS, vol. 3595, pp. 534–543. Springer, Heidelberg (2005)